A normative approach to neural computation: part 2

Dmitri "Mitya" Chklovskii

FLATIRON INSTITUTE
Division of Simons Foundation

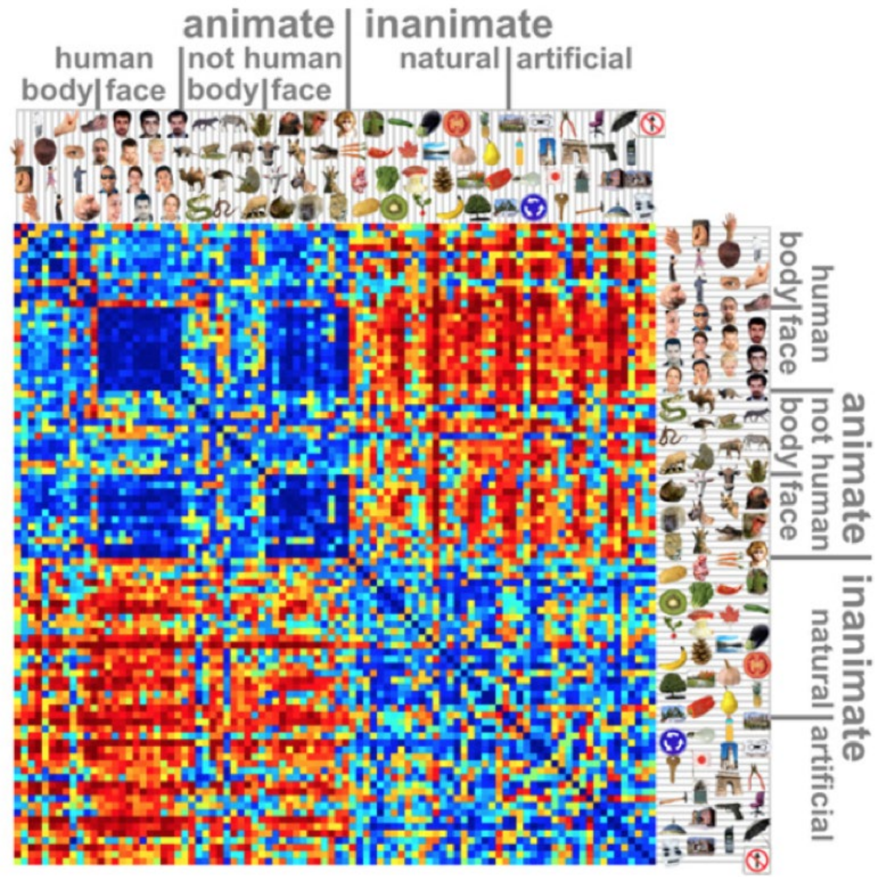
NYU NEUROSCIENCE

A normative approach to neural computation

- Reconstruction/decoding approach
  - Single-neuron PCA
  - Multiple neurons -> difficulties
- Similarity matching approach
  - Linear dimensionality reduction
  - Non-negative output
  - Two-compartment neurons

Similarity of neural activity patterns in IT

Human Cortex



Similarity

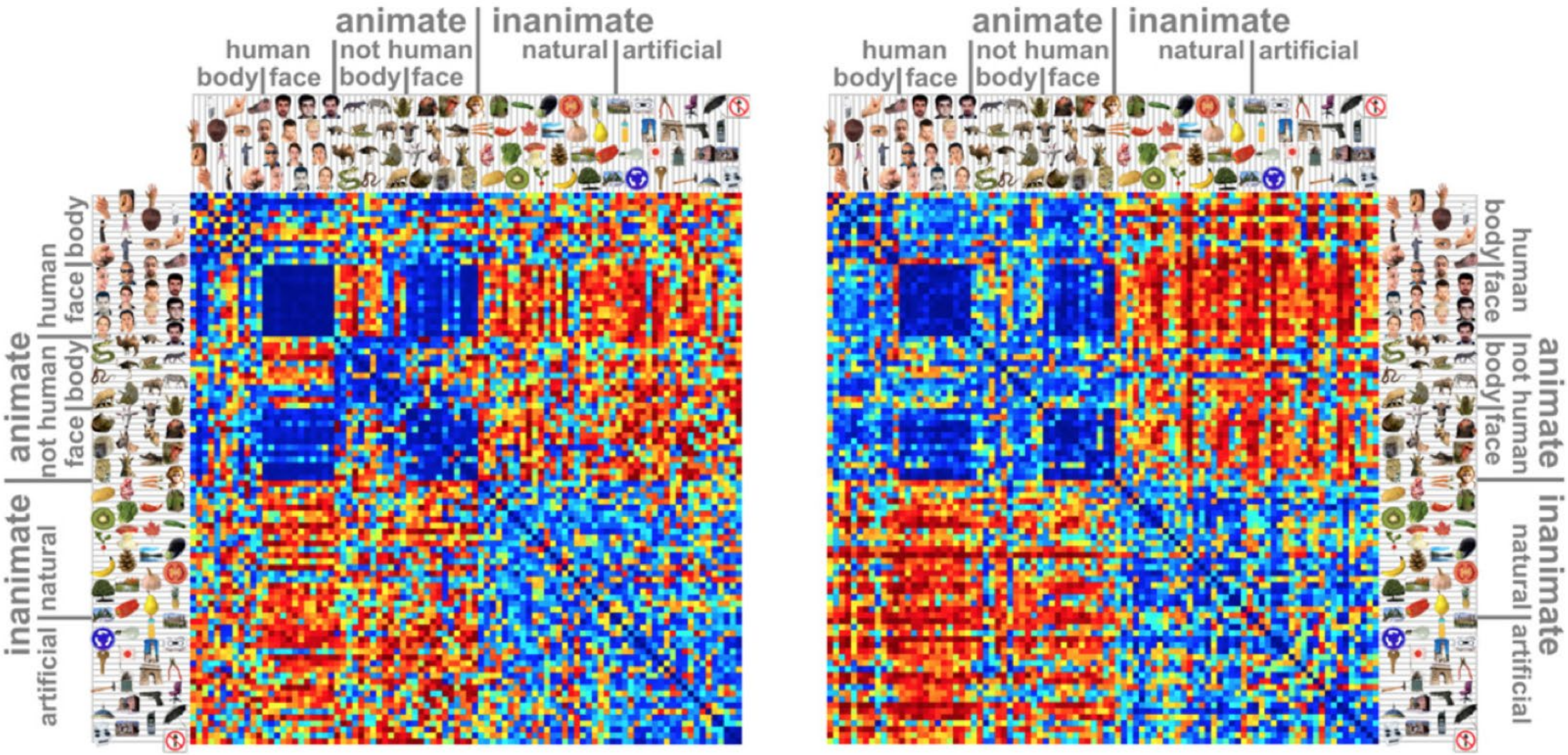
Kriegeskorte et. al., 2008

Kiani et. al., 2007

Similarity of neural activity patterns in IT

Monkey Cortex

Human Cortex



Similarity

Kriegeskorte et. al., 2008

Kiani et. al., 2007

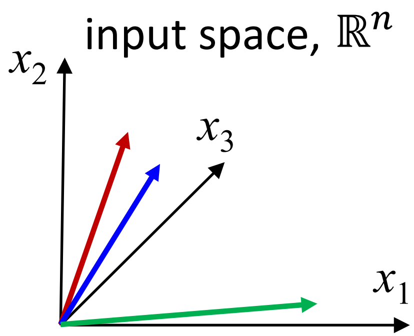
Neural representation is representation of similarities

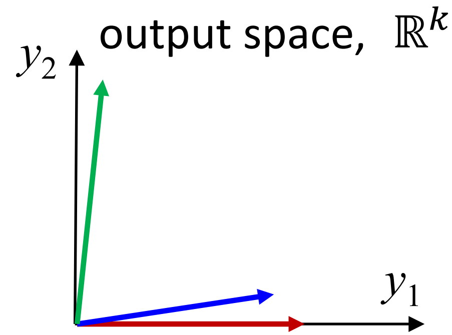
Shimon Edelman (1998)

Similarity matching: Similar input activity patterns evoke similar output activity patterns

Hu, Pehlevan, Chklovskii (2014)
Pehlevan, Chklovskii (2014)
Pehlevan, Hu, Chklovskii (2015)
Pehlevan, Chklovskii (2015)
Bahroun, Hunsicker, Soltoggio (2017)
Seung & Zung (2017)

Similar inputs produce similar outputs





Quantify similarity by the scalar product

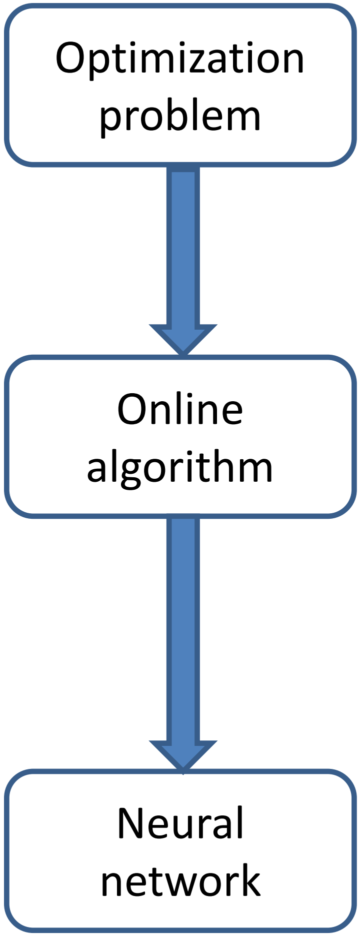
$$\boldsymbol{x}_1{}^{\top}\boldsymbol{x}_2 = \sum_{i=1}^{n} x_{i,1}x_{i,2} \qquad\qquad \boldsymbol{y}_1{}^{\top}\boldsymbol{y}_2 = \sum_{i=1}^{k} y_{i,1}y_{i,2}$$

$$\begin{pmatrix} \boldsymbol{x}_1{}^{\top}\boldsymbol{x}_1 & \boldsymbol{x}_1{}^{\top}\boldsymbol{x}_2 & \cdots & \boldsymbol{x}_1{}^{\top}\boldsymbol{x}_t \\ \boldsymbol{x}_2{}^{\top}\boldsymbol{x}_1 & \boldsymbol{x}_2{}^{\top}\boldsymbol{x}_2 & \cdots & \boldsymbol{x}_2{}^{\top}\boldsymbol{x}_t \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{x}_t{}^{\top}\boldsymbol{x}_1 & \boldsymbol{x}_t{}^{\top}\boldsymbol{x}_2 & \cdots & \boldsymbol{x}_t{}^{\top}\boldsymbol{x}_t \end{pmatrix} \approx \begin{pmatrix} \boldsymbol{y}_1{}^{\top}\boldsymbol{y}_1 & \boldsymbol{y}_1{}^{\top}\boldsymbol{y}_2 & \cdots & \boldsymbol{y}_1{}^{\top}\boldsymbol{y}_t \\ \boldsymbol{y}_2{}^{\top}\boldsymbol{y}_1 & \boldsymbol{y}_2{}^{\top}\boldsymbol{y}_2 & \cdots & \boldsymbol{y}_2{}^{\top}\boldsymbol{y}_t \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{y}_t{}^{\top}\boldsymbol{y}_1 & \boldsymbol{y}_t{}^{\top}\boldsymbol{y}_2 & \cdots & \boldsymbol{y}_t{}^{\top}\boldsymbol{y}_t \end{pmatrix}$$

$$\min_{\boldsymbol{y}_t, \boldsymbol{y}_\tau} \sum_{t=1}^{T}\sum_{\tau=1}^{T}\left(\boldsymbol{x}_t \cdot \boldsymbol{x}_\tau - \boldsymbol{y}_t \cdot \boldsymbol{y}_\tau\right)^2$$

PCA

Deriving a neural network



$$\min_{\boldsymbol{y}_t} \sum_{t=1}^{T}\sum_{\tau=1}^{T}\left(\boldsymbol{x}_t\cdot\boldsymbol{x}_\tau - \boldsymbol{y}_t\cdot\boldsymbol{y}_\tau\right)^2$$

$$\sum_{t=1}^{T}\sum_{\tau=1}^{T}\boldsymbol{x}_t^\top\boldsymbol{x}_\tau\boldsymbol{y}_t^\top\boldsymbol{y}_\tau = \sum_{t=1}^{T}\boldsymbol{y}_t^\top\left(\sum_{\tau=1}^{T}\boldsymbol{y}_\tau\boldsymbol{x}_\tau^\top\right)\boldsymbol{x}_t = \sum_{t=1}^{T}\boldsymbol{y}_t^\top\tilde{\boldsymbol{W}}^{YX}\boldsymbol{x}_t$$

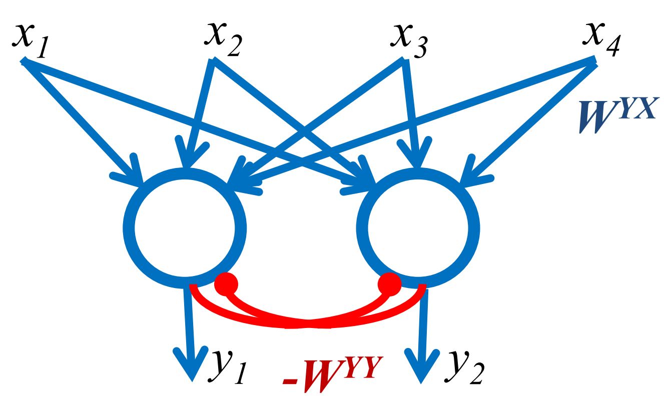
Iterate: $\boldsymbol{y}_t \leftarrow \boldsymbol{W}^{YX}\boldsymbol{x}_t - \boldsymbol{W}^{YY}\boldsymbol{y}_t$

$$W_{i,j,t}^{YX} = \frac{\sum_{\tau=1}^{t-1} y_{i,\tau}x_{j,\tau}}{\sum_{\tau=1}^{t-1} y_{i,\tau}^2}$$

$$W_{i,j,t}^{YX} \leftarrow W_{i,j,t-1}^{YX} + y_{i,t}\left(x_{j,t} - W_{i,j,t-1}^{YX}y_{i,t}\right)\Big/\sum_{\tau=1}^{t} y_{i,\tau}^2$$

$$W_{i,j,t}^{YY} \leftarrow W_{i,j,t-1}^{YY} + y_{i,t}\left(y_{j,t} - W_{i,j,t}^{YY}y_{i,t}\right)\Big/\sum_{\tau=1}^{t} y_{i,\tau}^2$$

Local learning rules!



A normative approach to neural computation

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  - Multiple neurons
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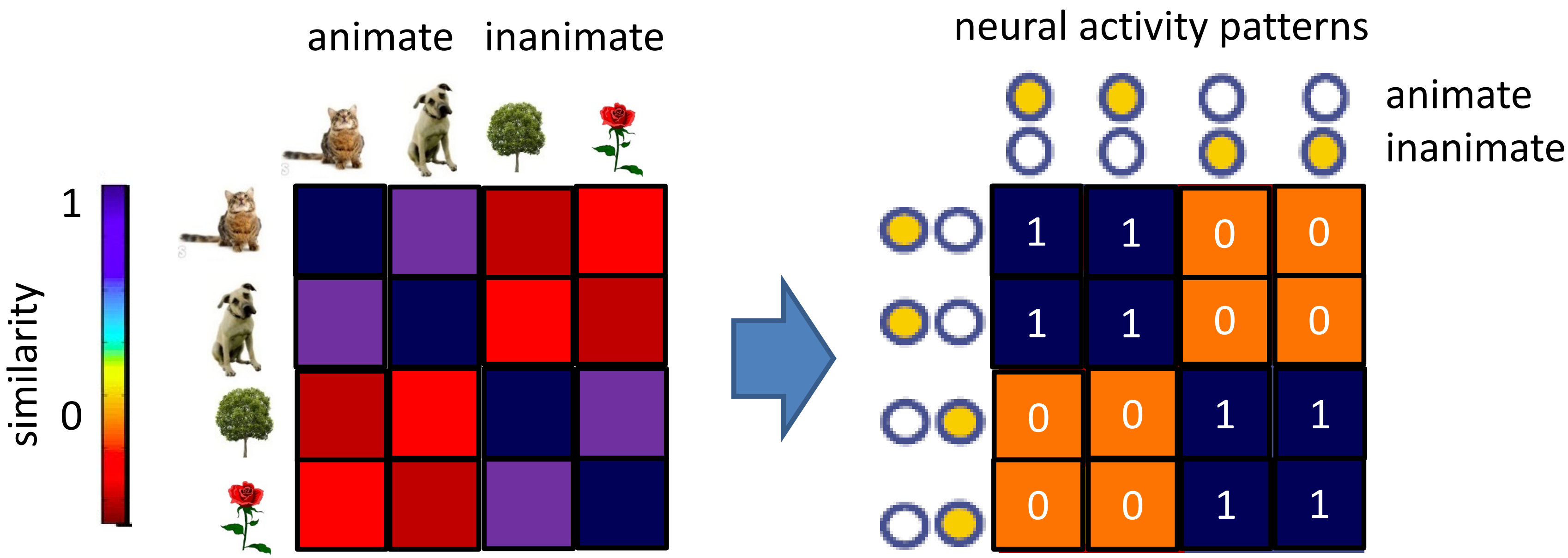
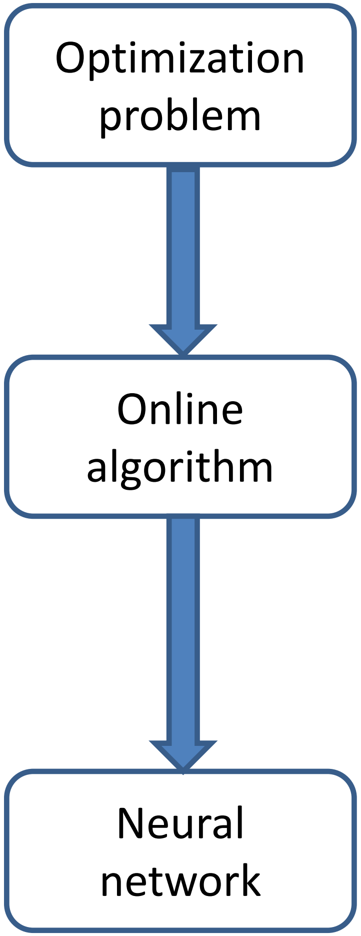
Non-negative output

$$\min_{y_t \geq 0} \sum_{t=1}^{T} \sum_{\tau=1}^{T} \left(\boldsymbol{x}_t \cdot \boldsymbol{x}_\tau - \boldsymbol{y}_t \cdot \boldsymbol{y}_\tau \right)^2$$



Deriving a neural network



$$\min_{\boldsymbol{y}_t \geq 0} \sum_{t=1}^{T} \sum_{\tau=1}^{T} \left(\boldsymbol{x}_t \cdot \boldsymbol{x}_\tau - \boldsymbol{y}_t \cdot \boldsymbol{y}_\tau \right)^2$$

$$\sum_{t=1}^{T} \sum_{\tau=1}^{T} \boldsymbol{x}_t^\top \boldsymbol{x}_\tau \boldsymbol{y}_t^\top \boldsymbol{y}_\tau = \sum_{t=1}^{T} \boldsymbol{y}_t^\top \left(\sum_{\tau=1}^{T} \boldsymbol{y}_\tau \boldsymbol{x}_\tau^\top \right) \boldsymbol{x}_t = \sum_{t=1}^{T} \boldsymbol{y}_t^\top \tilde{\boldsymbol{W}}^{YX} \boldsymbol{x}_t$$

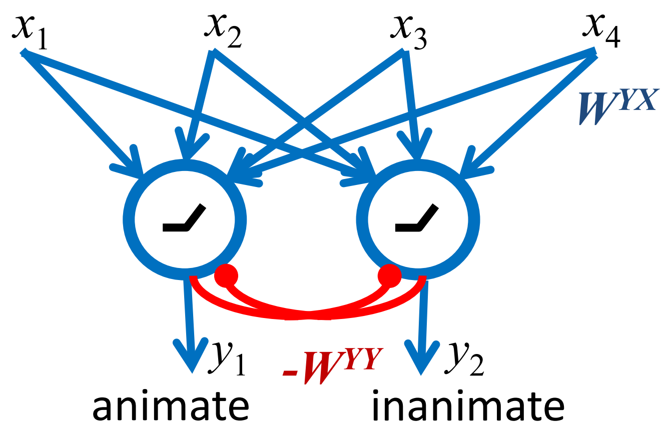
Iterate: $\boldsymbol{y}_t \leftarrow \max\left(\boldsymbol{W}^{YX} \boldsymbol{x}_t - \boldsymbol{W}^{YY} \boldsymbol{y}_t, 0 \right)$

$$W_{i,j,t}^{YX} = \frac{\sum_{\tau=1}^{t-1} y_{i,\tau} x_{j,\tau}}{\sum_{\tau=1}^{t-1} y_{i,\tau}^2}$$

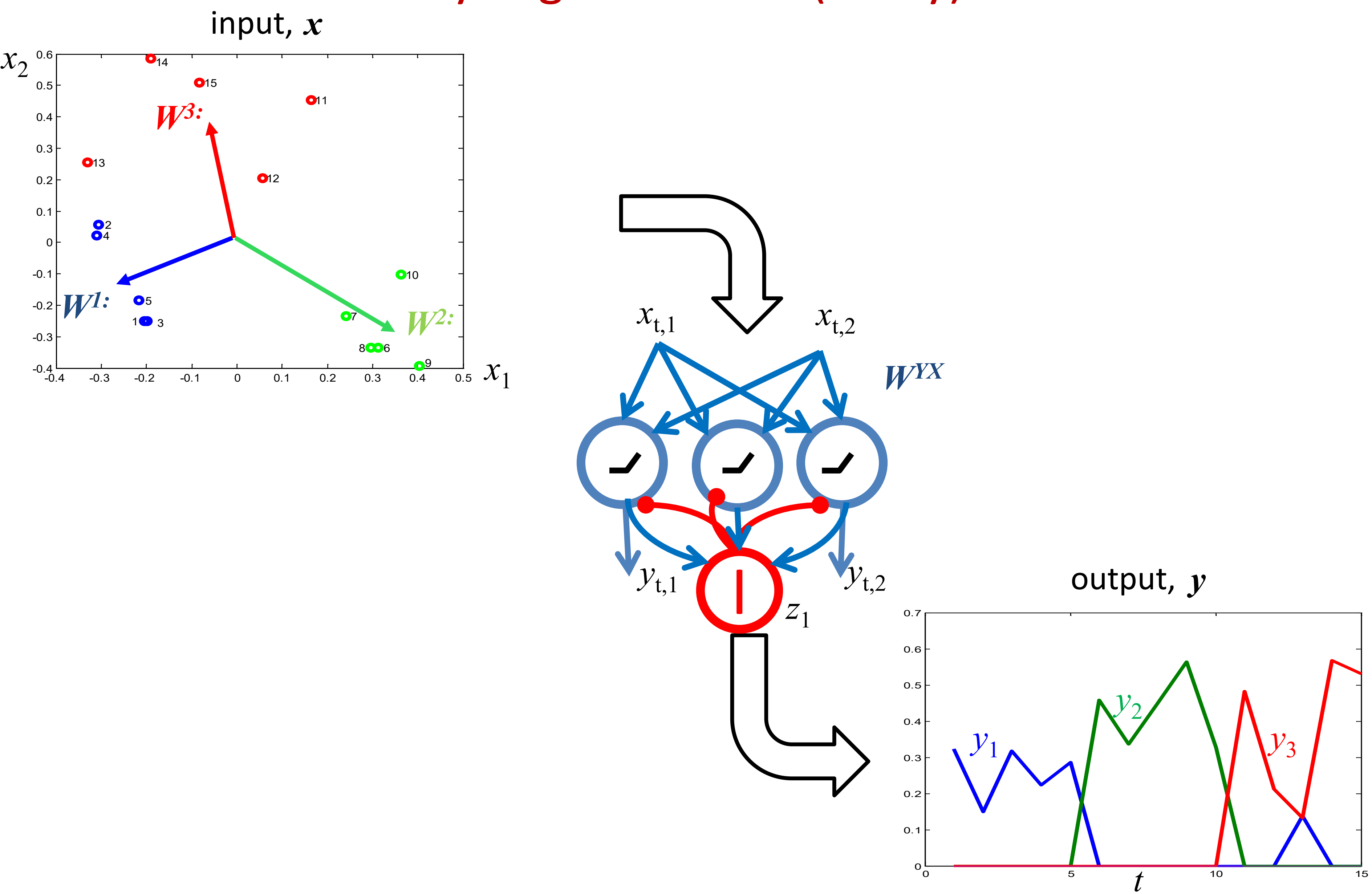
$$W_{i,j,t}^{YX} \leftarrow W_{i,j,t-1}^{YX} + y_{i,t} \left(x_{j,t} - W_{i,j,t-1}^{YX} y_{i,t} \right) \Big/ \sum_{\tau=1}^{t} y_{i,\tau}^2$$

$$W_{i,j,t}^{YY} \leftarrow W_{i,j,t-1}^{YY} + y_{i,t} \left(y_{j,t} - W_{i,j,t}^{YY} y_{i,t} \right) \Big/ \sum_{\tau=1}^{t} y_{i,\tau}^2$$

Local learning rules!

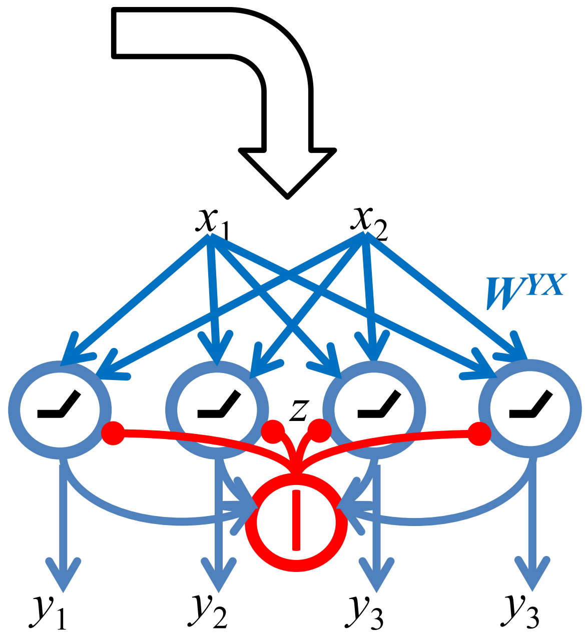


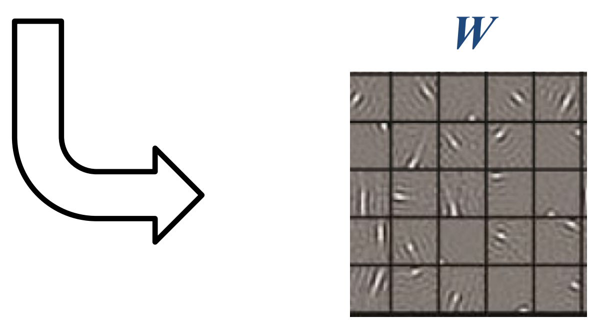
Similarity alignment can (softly) cluster



Similarity alignment network learns V1 features from natural images

natural images





Pehlevan & Chklovskii (2014)

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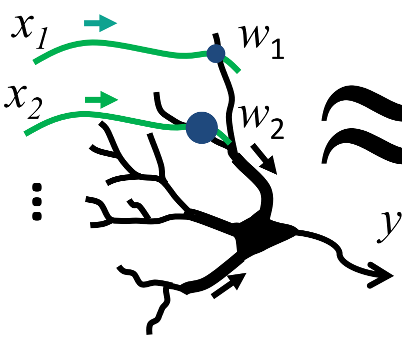
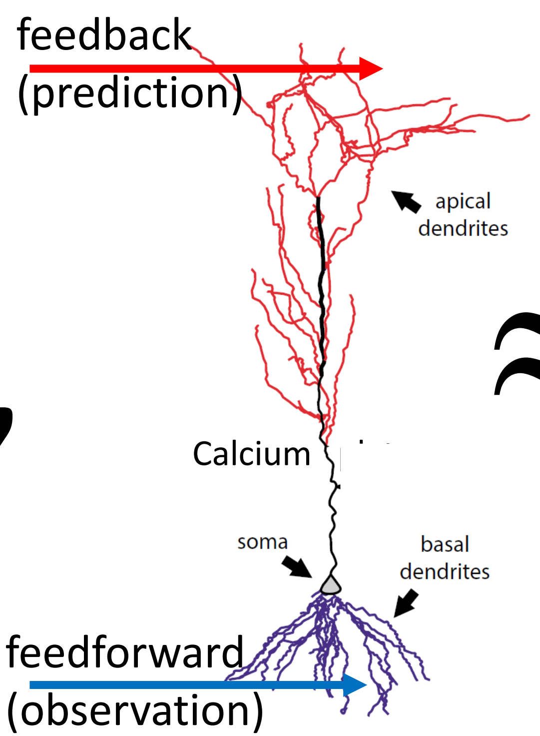
Similarity matching approach – 2 compartment neurons

A pyramidal neuron: 2 electrotonic compartments



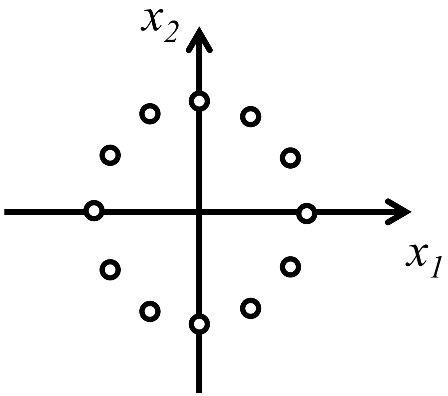
≈PCA,



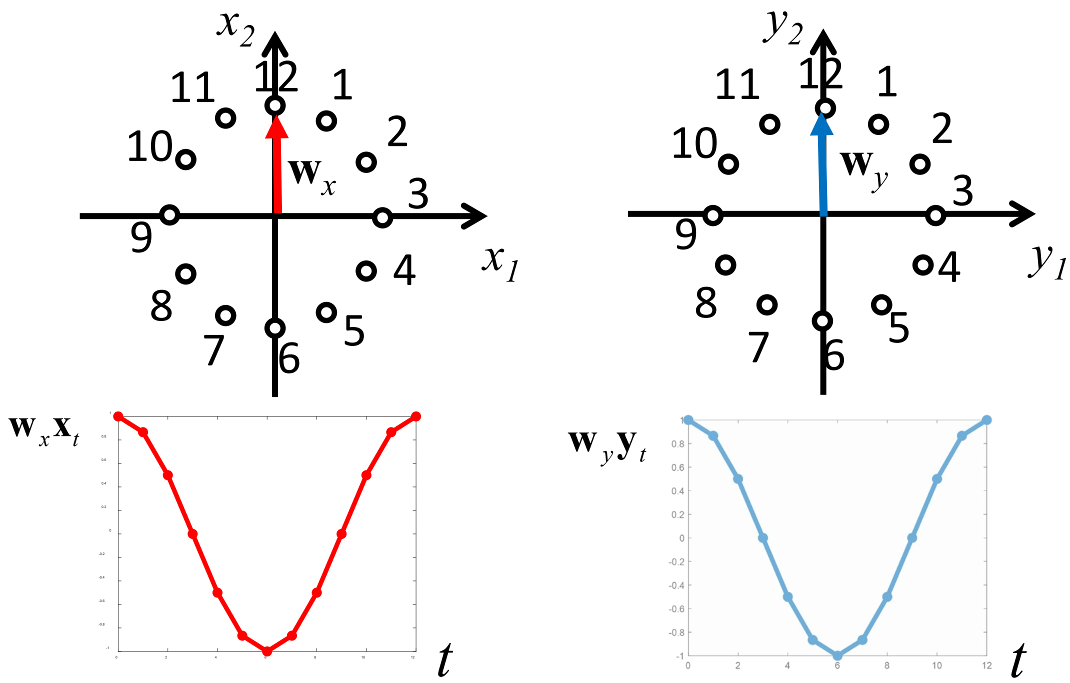
≈CCA

Schiller, Larkum, Spruston, Magee…
Mel, Poirazzi, Senn, Haga, Fukai, Machens, Deneve…

Canonical Correlation Analysis (CCA)

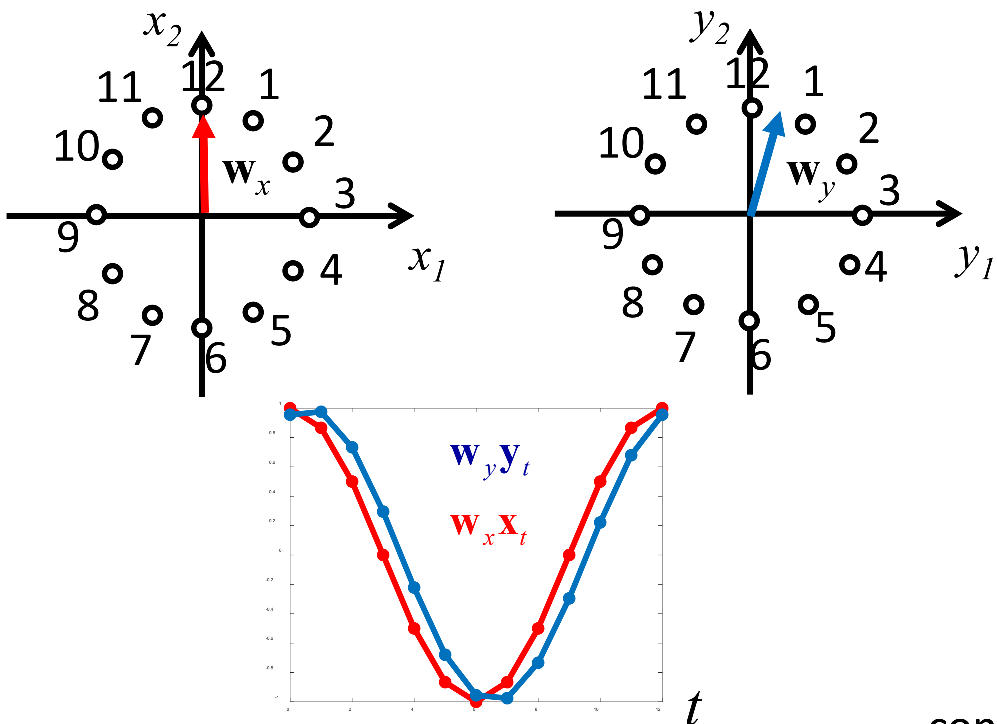


Canonical Correlation Analysis (CCA)



$$\max_{\mathbf{w}_x,\mathbf{w}_y} \sum_{t=1}^{T} \mathbf{w}_x \mathbf{x}_t \mathbf{y}_t^{\mathrm{T}} \mathbf{w}_y^{\mathrm{T}} \qquad \text{subject to} \qquad \frac{1}{T}\sum_{t=1}^{T} \mathbf{w}_x \mathbf{x}_t \mathbf{x}_t^{\mathrm{T}} \mathbf{w}_x^{\mathrm{T}} = 1, \qquad \frac{1}{T}\sum_{t=1}^{T} \mathbf{w}_y \mathbf{y}_t \mathbf{y}_t^{\mathrm{T}} \mathbf{w}_y^{\mathrm{T}} = 1$$

CCA learning rule



$$\arg\min_{\mathbf{w}_x,\mathbf{w}_y}\sum_{t=1}^{T}\left(\mathbf{w}_x\mathbf{x}_t-\mathbf{w}_y\mathbf{y}_t\right)\left(\mathbf{w}_x\mathbf{x}_t-\mathbf{w}_y\mathbf{y}_t\right)^{\mathrm{T}}=\arg\min_{\mathbf{w}_x,\mathbf{w}_y}-2\sum_{t=1}^{T}\mathbf{w}_x\mathbf{x}_t\mathbf{y}_t^{\mathrm{T}}\mathbf{w}_y^{\mathrm{T}}+\overbrace{\sum_{t=1}^{T}\mathbf{w}_x\mathbf{x}_t\mathbf{x}_t^{\mathrm{T}}\mathbf{w}_x^{\mathrm{T}}}^{\text{const}}+\overbrace{\sum_{t=1}^{T}\mathbf{w}_y\mathbf{y}_t\mathbf{y}_t^{\mathrm{T}}\mathbf{w}_y^{\mathrm{T}}}^{\text{const}}$$

$$=\arg\max_{\mathbf{w}_x,\mathbf{w}_y}\sum_{t=1}^{T}\mathbf{w}_x\mathbf{x}_t\mathbf{y}_t^{\mathrm{T}}\mathbf{w}_y^{\mathrm{T}}$$

Gradient descent: $\Delta\mathbf{w}_y=\eta\left(\mathbf{w}_x\mathbf{x}_T-\mathbf{w}_y\mathbf{y}_T\right)\mathbf{y}_T^{\mathrm{T}}$

Similarity matching approach – 2 compartment neurons

Pyramidal neuron as a Canonical Correlation Analyzer

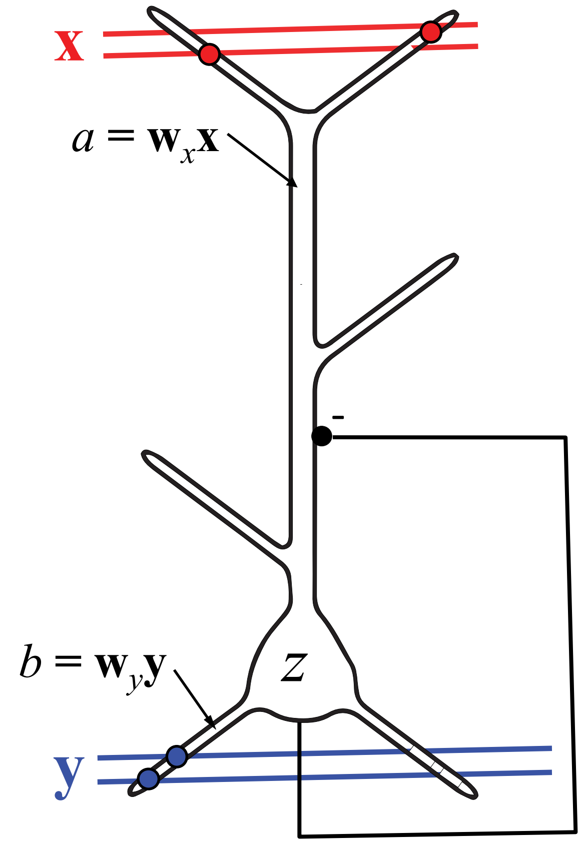
$$\Delta \mathbf{w}_y = \eta\left(\mathbf{w}_x \mathbf{x}_T - \mathbf{w}_y \mathbf{y}_T\right)\mathbf{y}_T{}^{\mathrm{T}} = \eta\left(a - b\right)\mathbf{y}_T{}^{\mathrm{T}}$$

$$z = a + b - z \;\longrightarrow\; z = (a+b)/2$$

$$\begin{aligned}\Delta \mathbf{w}_y &= \eta(2a - (a+b))\mathbf{y}_T{}^{\mathrm{T}} = \eta(2a - 2z)\mathbf{y}_T{}^{\mathrm{T}} \\ &= 2\eta(a - z)\mathbf{y}_T{}^{\mathrm{T}}\end{aligned}$$

Calcium plateau | Pre-synaptic spikes

Local non-Hebbian learning rule



Experiments: Golding et al., 2002; Bittner et al., 2017; Greinberger & Magee, 2020

Related algorithms: Lai & Fyfe, 1999; Pehlevan, Zhao, Sengupta & C, 2020

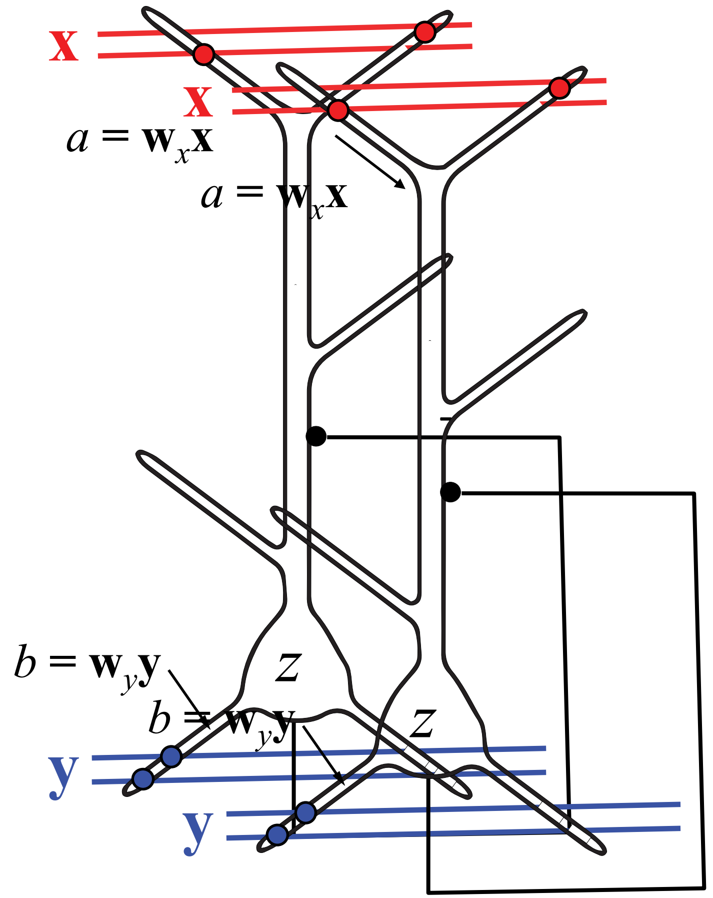
Multi-channel CCA

$$\mathbf{X} = [\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_T]; \qquad \mathbf{Y} = [\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_T];$$

$$\mathbf{V}_x = [\mathbf{w}_{x1}^{\mathrm{T}}, \mathbf{w}_{x2}^{\mathrm{T}}, \ldots, \mathbf{w}_{xk}^{\mathrm{T}}]^{\mathrm{T}}; \qquad \mathbf{V}_y = [\mathbf{w}_{y1}^{\mathrm{T}}, \mathbf{w}_{y2}^{\mathrm{T}}, \ldots, \mathbf{w}_{yk}^{\mathrm{T}}]^{\mathrm{T}};$$

$$\max_{\mathbf{V}_x, \mathbf{V}_y} \mathrm{Tr}\, \mathbf{V}_x \mathbf{X} \mathbf{Y}^{\mathrm{T}} \mathbf{V}_y^{\mathrm{T}} \qquad \text{subject to}$$

$$\mathbf{V}_x \mathbf{X} \mathbf{X}^{\mathrm{T}} \mathbf{V}_x^{\mathrm{T}} = I, \qquad \mathbf{V}_y \mathbf{Y} \mathbf{Y}^{\mathrm{T}} \mathbf{V}_y^{\mathrm{T}} = I,$$



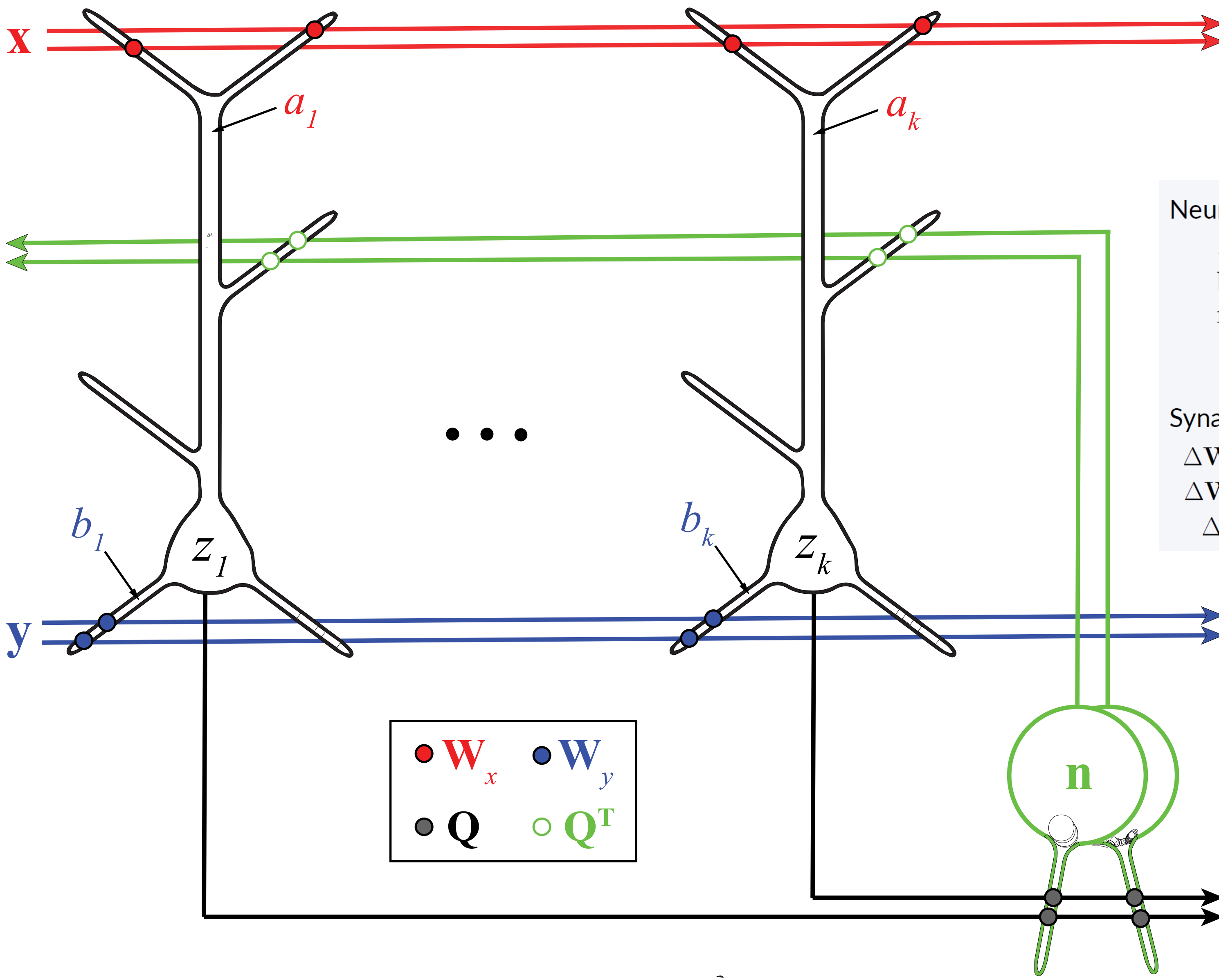
A similarity matching objective for the CCA neural network

$$\min_{\mathbf{Z}\in\mathbb{R}^{k\times T}} \max_{\mathbf{N}\in\mathbb{R}^{k\times T}} \frac{1}{T^2} \mathrm{Tr}\Big[\alpha T\, \mathbf{Z}^\top\mathbf{Z} - \mathbf{Z}^\top\mathbf{Z}\mathbf{X}^\top\mathbf{\Sigma}_{XX}^{-1}\mathbf{X}$$
$$-\mathbf{Z}^\top\mathbf{Z}\mathbf{Y}^\top\mathbf{\Sigma}_{YY}^{-1}\mathbf{Y} + \mathbf{N}^\top\mathbf{N}\left(\mathbf{Z}^\top\mathbf{Z} - T\mathbf{I}_T\right)\Big],$$

Optimization problem → Online algorithm → Neural network

Lipshutz, Bahroun, Golkar, Sengupta & C (2021)

Pyramidal network for multi-channel CCA



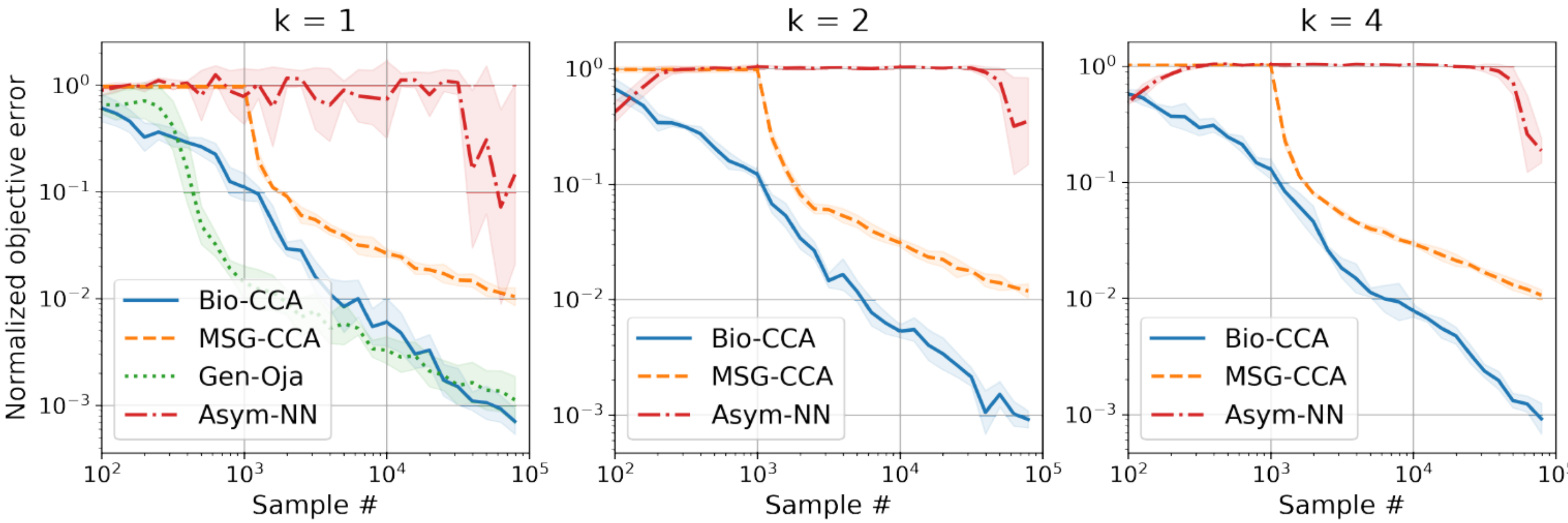
Neural dynamics:

$$\begin{aligned}
\mathbf{a}_t &= \mathbf{W}_x \mathbf{x}_t, \\
\mathbf{b}_t &= \mathbf{W}_y \mathbf{y}_t \\
\mathbf{n}_t &= \mathbf{Q}\mathbf{z}_t \\
\mathbf{z}_t &= \mathbf{a}_t + \mathbf{b}_t - \mathbf{Q}^\top \mathbf{n}_t
\end{aligned}$$

Synaptic updates:

$$\begin{aligned}
\Delta \mathbf{W}_x &= 2\eta_t(\mathbf{b}_t - \mathbf{Q}^\top \mathbf{n}_t)\mathbf{x}_t^\top \\
\Delta \mathbf{W}_y &= 2\eta_t(\mathbf{a}_t - \mathbf{Q}^\top \mathbf{n}_t)\mathbf{y}_t^\top \\
\Delta \mathbf{Q} &= \eta_t(\mathbf{n}_t \mathbf{z}_t^\top - \mathbf{Q}).
\end{aligned}$$

Comparison with other algorithms



Lipshutz, Bahroun, Golkar, Sengupta & C (2021)

Family of similarity matching networks

BIOLOGICAL FEATURE	MATHEMATICAL CONSTRUCT
Hebbian plasticity	Similarity alignment
Neural rectification, local RFs	Nonnegativity constraint
Adaptive neural thresholds	Rank and sparsity regularizers
Anti-Hebbian inhibitory interneurons	Constrained output similarity matrix
2-compartment neuron	Canonical correlation analysis (CCA)
…	…

Hu, Pehlevan & C (2014), Pehlevan & C (2014), Pehlevan, Hu & C (2015), Pehlevan & C (2015), Pehlevan & C (2016), Pehlevan, Mohan & C (2017), Pehlevan, Sengupta & C (2017), Tepper, Sengupta & C (2017), Pehlevan, Genkin & C (2017), Bahroun, Hunsicker, Soltoggio (2017), Seung & Zung (2017), Sengupta, Tepper, Pehlevan, Genkin & C (2018), Minden, Pehlevan & C (2018), Pehlevan & C (2019), Lipshutz, Windolf, Golkar, & C (2020), Golkar, Lipshutz, Bahroun, Sengupta & C (2020), Lipshutz, Bahroun, Golkar, Sengupta & C (2021)

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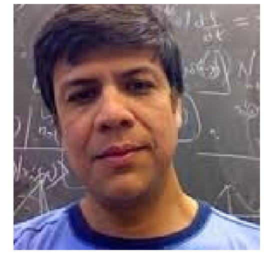
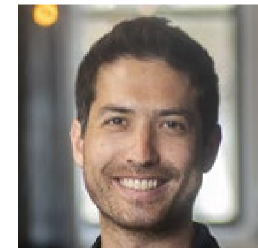
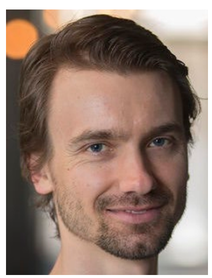


Anirvan Sengupta



David Lipshutz

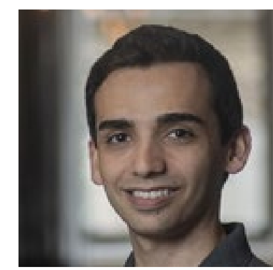
Victor Minden
-> Google



Nikolai Chapochnikov
->NYU

Tao Hu
->Texas A&M

Yanis Bahroun



Siavash Golkar