HW 2 soln key

1) This is answered in Oja's papers. You can also add together *n* increments to get that

$$w(n) - w(0) = \sum_{i=1}^{n} x(i)x(i)^T w(0) \propto Cw(0)$$

so that after *tn* iterations $w(tn) - w(0) \propto C^t w(0)$. Thus,

$$w(tn) \propto w(0) + C^t w(0) = w(0) + (V\Lambda V^T)^t w(0) = w(0) + (V\Lambda^t V^T)w(0)$$

So long as the initial iterate has a projection onto the leading eigenvector of *C*, this will grow to be proportional to the leading eigenvector (the first column of *V*, since $\lambda_1^t \gg \lambda_2^t \gg ... \gg \lambda_n^t$ and $\Lambda = \text{diag}(\lambda_i)$).

2.1) This is pretty easy.

2.2) Step 1: Consider the quantity inside the expectation. For a dataset of size n, the matrix *K* formed from that will be an outer product and thus positive definite. Step 2: The expectation of a random matrix which is positive definite matrix is positive definite.

3.1) Check the definitions directly. Without dividing by the eigenvalues, this is just the l2 inner product, and the eigenvalues are all positive.

3.2) $\|f\|_{\mathcal{H}} = \sqrt{\langle f, f\rangle_{\mathcal{H}}}$

3.3) This is more of an analysis question. Expand the function *f* in the kernel eigenbasis. The goal is to show that

$$f(x) = \langle K_x, f\rangle_{\mathcal{H}} = \langle K(x,\cdot), f(\cdot)\rangle_{\mathcal{H}} = \int K(x,x')f(x')d\mu(x).$$

Given that the eigenbasis is a complete orthonormal set for $L^2(\mu)$, the result follows after you plug in the expansions of kernel and *f*. Also, you have to check that K_x is in the RKHS, i.e. it has finite RKHS norm.

4) Check that the orthonormal bases I've given work. Then, use the norm formulae to get that

$$\|f\|_{\text{prod}}^2 = (\lambda_1 \lambda_0^{d-1})^{-1} = O(C^d) \text{ and } \|f\|_{\text{add}}^2 = \lambda_1^{-1} + \sum_{i=2}^{d} \lambda_0^{-1} = O(d)$$

These are the kind of kernels that you get with sparsely connected random networks! See: https://openreview.net/forum?id=rylt7mFU8S